Quantum Mechanics Exam Spring 2012

DISCLAIMER: This sample PhD qualifier exam only demonstrates the typical level of question that may be posed by the GRASP. Students must not infer anything regarding the content of their exam based on these examples; questions may be drawn from the full range of the topic.

Solve 4 out of the following 5 problems

Problem 1. For the potential

\[ V(x) = \begin{cases} 
0 & \text{if } x < 0 \\
-V_0 & \text{if } x \geq 0 
\end{cases} \]

calculate the reflection and transmission coefficients for a particle incident from the left as a function of the energy \( E \), with \( E > 0 \).

Problem 2. Consider the Hermitian operator \( \hat{B} \) corresponding to the observable \( b \).

(a) Show that the uncertainty in \( b \) can be written as \( \Delta b = \left( \langle b^2 \rangle - \langle b \rangle^2 \right)^{1/2} \).

(b) Show that if the wavefunction \( \varphi_b \) is an eigenfunction of \( \hat{B} \), then the uncertainty in the observable \( b \) is \( \Delta b = 0 \).

Problem 3. The energy of an electron trapped in a two-dimensional square corral with widths \( L_x = L_y = L \) and infinite walls is given by

\[ E(n_x, n_y) = \frac{\hbar^2}{8m_e} \left( \frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} \right) = \frac{\hbar^2}{8m_e L^2} (n_x^2 + n_y^2). \]

Seven electrons are all confined to this corral. Taking electron spin into account, construct the electron configuration for the ground state of the system of seven electrons. Assume that the electrons do not interact electrically or magnetically with each other.

(a) Draw an energy level diagram to scale that shows the lowest 5 energy levels for the square corral and indicate the quantum numbers \( n_x \) and \( n_y \) for each level.

(b) What is the degeneracy of each of the 5 lowest levels?

(c) In the ground state of the system of seven electrons, show which levels are occupied by how many electrons, and what their spin states are relative to each other (use up and down arrows). Give your reasoning!

(d) What is the total energy of this seven-electron system in its ground state as a multiple of \( \hbar^2/8m_e L^2 \)?
Problem 4. The Hamiltonian of the one-dimensional simple harmonic oscillator with mass $m$ and spring constant $k = m\omega^2$ is

$$H = \frac{1}{2m} [p^2 + (m\omega x)^2].$$

(a) Show that this Hamiltonian can be written also in the form

$$H = \hbar \omega (a_+ a_- + \frac{1}{2})$$

using the ladder operators $a_\pm = \frac{1}{\sqrt{2m\hbar\omega}} (\mp ip + m\omega x)$.

The ladder operators have the properties $a_+ \psi_n(x) = \sqrt{n+1} \psi_{n+1}(x)$ and $a_- \psi_n(x) = \sqrt{n} \psi_{n-1}(x)$, where $\psi_n(x)$ is the $n$-th eigenstate ($n=0,1,2,...$) of the Hamiltonian with eigenvalue $E_n = (n + \frac{1}{2}) \hbar \omega$.

(b) What do you get if you apply the step-down operator to the ground state?

(c) Demonstrate that the energy of the ground state is indeed $E_0 = \frac{1}{2} \hbar \omega$.

(d) Calculate the expectation value $\langle x \rangle$ of the position $x$ of the mass $m$ in the ground state $\psi_0$.

(e) The wave function for the ground state is $\psi_0(x) = A e^{\frac{-m\omega}{2\hbar} x^2}$ where $A$ is some normalization constant. Find the wave function $\psi_1(x)$ for the first excited state of the simple harmonic oscillator (no need to normalize it) and sketch it.

Problem 5. Consider a particle of mass $m$ in an infinite one-dimensional square well of width $a$ with normalized eigenfunctions and energies given by

$$\varphi_n = \sqrt{\frac{2}{a}} \sin \left(\frac{n\pi x}{a}\right), \quad E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}, \quad n = 1, 2, 3, ...$$

Suppose at time $t = 0$, the wavefunction for the system is $\Psi = C (\varphi_1 + 2\varphi_2 + 3\varphi_3)$

(a) Normalize $\Psi$.

(b) Calculate the expectation value of the energy. How does your answer depend upon time?

(c) At time $t = 10s$, what will the normalized wavefunction be?