Name ____ MTH 2001 - Calculus 3

PRACTICE ADVANCED STANDING EXAM

No outside resources are permitted including: notes, textbooks, cell phones or any other electronics. Show all work. Solutions without explanations will receive no points. Simplify your answers. Circle your final answers.

1. Calculate the given quantities using the following vectors:

$\vec{\mathbf{u}} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\vec{\mathbf{v}} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

a. $2\vec{\mathbf{u}} + 3\vec{\mathbf{v}}$

$2(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) + 3(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + 9\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}} = 11\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 1\hat{\mathbf{k}}$

b. $|\vec{\mathbf{u}}|$ and $|\vec{\mathbf{v}}|$

$|\vec{\mathbf{u}}| = \sqrt{(1)^2 + (1)^2 + (-2)^2} = \sqrt{1+1+4} = \sqrt{6}$

$|\vec{\mathbf{v}}| = \sqrt{(3)^2 + (-2)^2 + (1)^2} = \sqrt{9+4+1} = \sqrt{14}$

c. A unit vector going in the same direction as $\vec{\mathbf{u}}$

$\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} = \frac{1}{\sqrt{6}}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = \frac{1}{\sqrt{6}}\hat{\mathbf{i}} + \frac{1}{\sqrt{6}}\hat{\mathbf{j}} - \frac{2}{\sqrt{6}}\hat{\mathbf{k}}$

d. $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$

$(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \cdot (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = (1)(3) + (1)(-2) + (-2)(1) = 3 - 2 - 2 = -1$

e. $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$

$(\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \times (3\mathbf{i} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \mathbf{1} & \mathbf{1} & \mathbf{-2} \\ \mathbf{3} & \mathbf{-2} & \mathbf{1} \end{vmatrix} = +\hat{\mathbf{i}}\begin{vmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{-2} & \mathbf{1} \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{3} & \mathbf{1} \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{3} & \mathbf{-2} \end{vmatrix}$

$= +\hat{\mathbf{i}}(1-4) - \hat{\mathbf{j}}(1+6) + \hat{\mathbf{k}}(-2-3) = -3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

f. Find the angle between the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$

$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = |\vec{\mathbf{u}}||\vec{\mathbf{v}}| \cos\theta$

$[-1] = [\sqrt{6}][\sqrt{14}] \cos\theta$

So $\cos\theta = -\frac{1}{\sqrt{6}\cdot\sqrt{14}}$, which means that $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{6}\cdot\sqrt{14}}\right) = \cos^{-1}\left(-\frac{1}{2\sqrt{21}}\right)$

g. $\text{comp}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$

$\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{|\vec{\mathbf{u}}|} = \frac{-1}{\sqrt{6}}$

h. $\text{proj}_{\vec{\mathbf{u}}}\vec{\mathbf{v}}$

$\left(\frac{\vec{\mathbf{u}}\cdot\vec{\mathbf{v}}}{|\vec{\mathbf{u}}|}\right)\frac{\vec{\mathbf{u}}}{|\vec{\mathbf{u}}|} = \frac{-1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\hat{\mathbf{i}} + \frac{1}{\sqrt{6}}\hat{\mathbf{j}} - \frac{2}{\sqrt{6}}\hat{\mathbf{k}}\right) = -\frac{1}{6}\hat{\mathbf{i}} - \frac{1}{6}\hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}}$

2. If $\vec{\mathbf{r}}(t) = \langle t^3, \frac{1}{\pi}\sin(\pi t), 4 + 2\ln t\rangle$, find the equation of the tangent line to the curve at the point when $t = 1$

At $t = 1$, If $\vec{\mathbf{r}}(t) = \langle 1^3, \frac{1}{\pi}\sin(\pi), 4 + 2\ln 1\rangle = \langle 1,0,4\rangle$

Now $\vec{\mathbf{r}'}(t) = \langle 3t^2, \cos(\pi t), \frac{2}{t}\rangle$, which means that $\vec{\mathbf{r}'}(1) = \langle 3(1)^2, \cos(\pi), \frac{2}{1}\rangle = \langle 3,-1,2\rangle$

Therefore, the equations of the tangent line are:

$x(t) = 1 + 3t$ and $y(t) = 0 - 1t$ and $z(t) = 4 + 2t$

Or, alternately, the tangent line can be written as: $\langle 1+3t, -t, 4+2t\rangle$

3. Find the arclength of the curve $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, t\rangle$ between the endpoints where $t = \pi$ and $t = 4\pi$

Since $\vec{\mathbf{r}}(t) = \langle 3\cos t, 3\sin t, t\rangle$, then $\vec{\mathbf{r}}'(t) = \langle -3\sin t, 3\cos t, 1\rangle$
Then the arclength is given by: $\int_{\pi}^{4\pi} \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (1)^2}dt = \int_{\pi}^{4\pi} \sqrt{9\sin^2 t + 9\cos^2 t + 1}dt$
$\int_{\pi}^{4\pi} \sqrt{9(\sin^2 t + \cos^2 t) + 1}dt = \int_{\pi}^{4\pi} \sqrt{9+1}dt = \int_{\pi}^{4\pi} \sqrt{10}dt = t\sqrt{10}\Big|_{t=\pi}^{t=4\pi} = 4\pi\sqrt{10} - \pi\sqrt{10} = 3\pi\sqrt{10}$

4. [14 pts] Evaluate the following limit, or explain why it does not exist: $\lim_{(x,y)\to(0,0)} \frac{10x\sin^2 y}{x^2+\sin^4 y}$

Along the path of the vertical line $y = 0$, the problem becomes: $\lim_{(x,0)\to(0,0)} \frac{10x(0)^2}{x^2+(0)^4} = \lim_{x\to 0} \frac{0}{x^2} = 0$
Along the path of the curve $x = \sin^2 y$, the problem becomes:
$\lim_{(\sin^2 y, y)\to(0,0)} \frac{10(\sin^2 y)\sin^2 y}{(\sin^2 y)^2+\sin^4 y} = \lim_{y\to 0} \frac{10\sin^4 y}{\sin^4 y+\sin^4 y} = \lim_{y\to 0} \frac{10\sin^4 y}{2\sin^4 y} = \frac{10}{2} = 5$
If a limit exists, it must be the same along *every* path. Since we have two paths that do not produce the same result, then the limit does not exist.

5. Use information about the gradient to answer the following about the surface

$$f(x,y) = x^2 + y^3 - 5xy$$

(a) What is the directional derivative at the point $(1,2)$ in the direction of the vector $\vec{\mathbf{v}} = \langle 3,4\rangle$

Since $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle = \langle 2x - 5y, 3y^2 - 5x\rangle$, then at the point $(1,2)$, we have $\nabla f(1,2) = \langle -8,7\rangle$
Now $\vec{\mathbf{v}} = \langle 3,4\rangle$ is not a unit vector since $|\vec{\mathbf{v}}| = \sqrt{3^2+4^2} = \sqrt{25} = 5$. Thus, we need to make a unit vector in the same direction: $\vec{\mathbf{w}} = \frac{\vec{\mathbf{v}}}{|\vec{\mathbf{v}}|} = \langle \frac{3}{5}, \frac{4}{5}\rangle$.
Then the directional derivative is given by $\nabla f \cdot \vec{\mathbf{w}}$: $\langle -8,7\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \frac{-24}{5} + \frac{28}{5} = \frac{4}{5}$

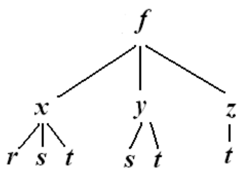
(b) What is the maximum value of the directional derivative?

The maximum value of the directional derivative is given by:
$|\nabla f| = |\langle -8,7\rangle| = \sqrt{(-8)^2 + (7)^2} = \sqrt{64+49} = \sqrt{113}$

(c) In what direction (given as a unit vector) is this largest directional derivative?

The largest directional derivative occurs in the direction of ∇f itself.
So we need to turn ∇f into a unit vector, which produces: $\frac{\nabla f}{|\nabla f|} = \langle -\frac{8}{\sqrt{113}}, \frac{7}{\sqrt{113}}\rangle$

6. Given the function $f(x, y, z) = x^2 + yz^3 + 2xy^2$, where $x = rs\sin t$ and $y = s^2 e^t$ and $z = 3t + 2$, find the value of the partial derivatives $\frac{\partial f}{\partial r}, \frac{\partial f}{\partial s}$, and $\frac{\partial f}{\partial t}$ at the point where $r = 1$, $s = 2$, and $t = 0$.



$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r}$$
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

If $r = 1, s = 2$, and $t = 0$, then $x = (1)(2)\sin(0) = 0$ and $y = (2)^2 e^{(0)} = 4$ and $z = 3(0) + 2 = 2$

Now $\frac{\partial f}{\partial x} = 2x + 2y^2 \Rightarrow \frac{\partial f}{\partial x} = 2(0) + 2(4)^2 = 32$
and $\frac{\partial f}{\partial y} = z^3 + 4xy \Rightarrow \frac{\partial f}{\partial y} = (2)^3 + 4(0)(4) = 8$
and $\frac{\partial f}{\partial z} = 3yz^2 \Rightarrow \frac{\partial f}{\partial z} = 3(4)(2)^2 = 48$

Furthermore:
$\frac{\partial x}{\partial r} = s\sin t \Rightarrow \frac{\partial x}{\partial r} = (2)\sin(0) = 0$
$\frac{\partial x}{\partial s} = r\sin t \Rightarrow \frac{\partial x}{\partial s} = (1)\sin(0) = 0$ $\qquad \frac{\partial y}{\partial s} = 2se^t \Rightarrow \frac{\partial y}{\partial s} = 2(2)e^{(0)} = 4$
$\frac{\partial x}{\partial t} = rs\cos t \Rightarrow \frac{\partial x}{\partial t} = (1)(2)\cos(0) = 2$ $\qquad \frac{\partial y}{\partial t} = s^2 e^t \Rightarrow \frac{\partial y}{\partial s} = (2)^2 e^{(0)} = 4$ $\qquad \frac{\partial z}{\partial t} = 3 \Rightarrow \frac{\partial z}{\partial t} = 3$

Thus: $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} = [32][0] = 0$
$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = [32][0] + [8][4] = 32$
$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = [32][2] + [8][4] + [48][3] = 64 + 32 + 144 = 240$

7. Find and classify all the Critical Points of the function $f(x,y) = 3xy - x^2y - xy^2$ as Relative Maxima, Relative Minima, or Saddle Points.

First, we identify any Critical Points. That is, we find where $f_x = 0$ and $f_y = 0$
$f_x = 3y - 2xy - y^2 \Rightarrow 3y - 2xy - y^2 = 0$
$f_y = 3x - x^2 - 2xy \Rightarrow 3x - x^2 - 2xy = 0$

If $3y - 2xy - y^2 = 0$, then this equation factors: $y(3 - 2x - y) = 0$,
meaning that either $y = 0$ or $3 - 2x - y = 0 \Rightarrow y = 3 - 2x$

If $y = 0$, then we have $3x - x^2 - 2x(0) = 0 \Rightarrow 3x - x^2 = 0$ from our second equation.
This factors into $x(3 - x) = 0$, producing two solution: $x = 0$ or $x = 3$
Therefore we get $(0,0)$ and $(3,0)$ as Critical Points that we need to check.

Alternately, if $y = 3 - 2x$, then the second equation becomes: $3x - x^2 - 2x(3 - 2x) = 0$
$3x - x^2 - 6x + 4x^2 = 0$
$3x^2 - 3x = 0$
This factors into $3x(x - 1) = 0$, which produces two solutions: $x = 0$ or $x = 1$
If $x = 0$, then $y = 3 - 2(0) = 3$
If $x = 1$, then $y = 3 - 2(1) = 1$
Thus we have two additional Critical Points to check as well: $(0,3)$ and $(1,1)$

We need to check all of these in the equation $D = (f_{xx})(f_{yy}) - (f_{xy})^2$
Since $f_{xx} = -2y$ and $f_{yy} = -2x$ and $f_{xy} = 3 - 2x - 2y$

At $(0,0)$, we have $f_{xx} = 0$ and $f_{yy} = 0$ and $f_{xy} = 3$, making $D = (0)(0) - (3)^2 = -9$
Since D is negative, this tells us that $(0,0)$ is a Saddle Point.

At $(3,0)$, we have $f_{xx} = 0$ and $f_{yy} = -6$ and $f_{xy} = -3$, making $D = (0)(-6) - (-3)^2 = -9$
Since D is negative, this tells us that $(3,0)$ is a Saddle Point.

At $(0,3)$, we have $f_{xx} = -6$ and $f_{yy} = 0$ and $f_{xy} = -3$, making $D = (-6)(0) - (-3)^2 = -9$
Since D is negative, this tells us that $(0,3)$ is a Saddle Point.

At $(1,1)$, we have $f_{xx} = -2$ and $f_{yy} = -2$ and $f_{xy} = -1$, making $D = (-2)(-2) - (-1)^2 = +3$
Since D is positive and f_{xx} is negative, this tells us that $(1,1)$ is a Relative Maximum.

8. Find the Absolute Max and Absolute Min of the function $f(x, y) = x^2 - 2xy + 2y$ on the triangular region in the *xy*-plane bounded by the points $(0,0)$ and $(2,0)$ and $(2,4)$

First, we identify any Critical Points that lie in the interior of the region. That is, we solve $f_x = 0$ and $f_y = 0$
$f_x = 0 \Rightarrow 2x - 2y = 0 \Rightarrow x = y$
$f_y = 0 \Rightarrow -2x + 2 = 0 \Rightarrow x = 1$
Since $x = 1$ and $x = y$, then $y = 1$. Thus the only Critical Point we have in the interior is $(1,1)$.
In addition, we know we will have to check the three endpoints where the edges connect, so this gives us three more points to add to our growing list: $(0,0)$ and $(2,0)$ and $(2,4)$

Now we need to check and see if there are any other points that we need to worry about that actually lie on the edges themselves. We begin with the bottom edge. It's a horizontal line, so its equation is $y = 0$.
Plugging this in to our original formula, that equation degenerates into: $f(x, 0) = x^2 - 2x(0) + 2(0) = x^2$
We look for Critical Points of this equation, places where $f' = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$
This only yields $x = 0$, which together with the restriction that $y = 0$ for this edge gives us the point $(0,0)$.
That point was already in our list of places to check, so we don't get any new points to worry about from this edge.

The right side of the triangle is a vertical line, given by the equation $x = 2$. Again, we plug that into the original formula, which forces it to degenerate into: $f(2, y) = (2)^2 - 2(2)y + 2y = 4 - 2y$
We look for Critical Points of this reduced formula but quickly realize there aren't any because the derivative here is always $f' = -2$, which means it can't ever be zero. So we don't pick up any new points from this edge either.

Finally, we come to the slanted edge of the triangle, which is given by the formula $y = 2x$. Plugging this into the original, the equation again degenerates into: $f(x, 2x) = x^2 - 2x(2x) + 2(2x) = -3x^2 + 4x$
Once again, we look for Critical Points of this new equation: $f' = -6x + 4$
Now $f' = 0 \Rightarrow -6x + 4 = 0 \Rightarrow x = \frac{2}{3}$ Finally, we have something new! When $x = \frac{2}{3}$, then $y = 2\left(\frac{2}{3}\right) = \frac{4}{3}$, so we need to add the point $\left(\frac{2}{3}, \frac{4}{3}\right)$ to our list.

So we have a total of five points to check by plugging them into our original function and evaluating:
$f(1,1) = (1)^2 - 2(1)(1) + 2(1) = 1 - 2 + 2 = 1$
$f(0,0) = (0)^2 - 2(0)(0) + 2(0) = 0 - 0 + 0 = 0$
$f(2,0) = (2)^2 - 2(2)(0) + 2(0) = 4 - 0 + 0 = 4$
$f(2,4) = (2)^2 - 2(2)(4) + 2(4) = 4 - 16 + 8 = -4$
$f\left(\frac{2}{3}, \frac{4}{3}\right) = \left(\frac{2}{3}\right)^2 - 2\left(\frac{2}{3}\right)\left(\frac{4}{3}\right) + 2\left(\frac{4}{3}\right) = \frac{4}{9} - \frac{16}{9} + \frac{8}{3} = \frac{4}{9} - \frac{16}{9} + \frac{24}{9} = \frac{12}{9} = \frac{4}{3} \approx 1.333\ldots$
Of all of these output values, -4 is the lowest, so it is the Abs Min.
4 is the highest, so it is the Abs Max.

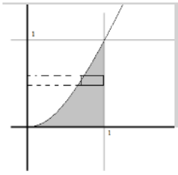
9. Find the Absolute Max and Absolute Min of the function $f(x,y) = x^2 + xy + y^2$ on the disk $x^2 + y^2 \le 8$.

First, we identify any Critical Points that lie in the interior of the region. That is, we solve $f_x = 0$ and $f_y = 0$
$f_x = 0 \Rightarrow 2x + y = 0$ and $f_y = 0 \Rightarrow x + 2y = 0$
Solving the first equation, we get: $y = -2x$
Plugging this in to the second equation, we get: $x + 2(-2x) = 0 \Rightarrow -3x = 0 \Longrightarrow x = 0$
Since $x = 0$, then $y = -2(0) = 0$ and so we have only one critical point in the interior, namely (0,0)
Next, we use Lagrange multipliers on the boundary.
$f = x^2 + xy + y^2 \Longrightarrow \nabla f = \langle 2x + y, x + 2y\rangle$
$g = x^2 + y^2 - 8 \Longrightarrow \nabla g = \langle 2x, 2y\rangle$
Now if $\nabla f = \lambda \nabla g$, then $\langle 2x + y, x + 2y\rangle = \lambda\langle 2x, 2y\rangle$ =$\langle 2x\lambda, 2y\lambda\rangle$
Thus $2x + y = 2x\lambda \Longrightarrow y = 2x\lambda - 2x \Longrightarrow y = 2x(\lambda - 1)$ from our first coordinate.
Likewise, from our second, we get $x + 2y = 2y\lambda \Longrightarrow x = 2y\lambda - 2y \Longrightarrow x = 2y(\lambda - 1)$
Substituting the first into the second, we get: $x = 2\big(2x(\lambda - 1)\big)(\lambda - 1) = 4x(\lambda - 1)^2$
Moving terms around: $4x(\lambda - 1)^2 - x = 0 \Longrightarrow x(4(\lambda - 1)^2 - 1) = 0$
Now, either $x = 0$, or $4(\lambda - 1)^2 - 1 = 0$
If $x = 0$, then $x^2 + y^2 = 8 \Longrightarrow 0 + y^2 = 8 \Longrightarrow y = \pm\sqrt{8}$, which gives us two new potential Critical Points to check, namely $\big(0, \sqrt{8}\big)$ and $\big(0, -\sqrt{8}\big)$
Alternately, if $4(\lambda - 1)^2 - 1 = 0 \Longrightarrow (\lambda - 1)^2 = \frac{1}{4} \Longrightarrow \lambda - 1 = \pm\frac{1}{2} \Longrightarrow \lambda = \frac{3}{2}$ or $\lambda = \frac{1}{2}$
If $\lambda = \frac{3}{2}$, then $y = 2x(\lambda - 1) \Longrightarrow y = x$ and therefore $x^2 + y^2 = 8 \Longrightarrow 2x^2 = 8 \Longrightarrow x = \pm 2$
This gives two more potential critical points: (2,2) and $(-2,-2)$
Finally, if $\lambda = \frac{1}{2}$, then $y = 2x(\lambda - 1) \Longrightarrow y = -x$ and therefore $x^2 + y^2 = 8 \Longrightarrow 2x^2 = 8 \Longrightarrow x = \pm 2$
This gives the final two potential critical points: $(2,-2)$ and $(-2,2)$

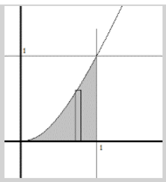
We have seven points to check. The results are as follows:
$f(0,0) = (0)^2 + (0)(0) + (0)^2 = 0 + 0 + 0 = 0$
$f\big(0, \sqrt{8}\big) = (0)^2 + (0)\big(\sqrt{8}\big) + \big(\sqrt{8}\big)^2 = 0 + 0 + 8 = 8$
$f\big(0, -\sqrt{8}\big) = (0)^2 + (0)\big(-\sqrt{8}\big) + \big(-\sqrt{8}\big)^2 = 0 + 0 + 8 = 8$
$f(2,2) = (2)^2 + (2)(2) + (2)^2 = 4 + 4 + 4 = 12$
$f(-2,2) = (-2)^2 + (-2)(2) + (2)^2 = 4 - 4 + 4 = 4$
$f(2,-2) = (2)^2 + (2)(-2) + (-2)^2 = 4 - 4 + 4 = 4$
$f(-2,-2) = (-2)^2 + (-2)(-2) + (-2)^2 = 4 + 4 + 4 = 12$
The Absolute Max is 12. The Absolute Min is 0.

10. Evaluate the following integral by reversing the order of integration: $\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3} dxdy$

Since the integral is: $\int_{y=0}^{y=1} \int_{x=\sqrt{y}}^{x=1} \frac{ye^{x^2}}{x^3} dxdy$, then the limits are clearly $y = 0$, $y = 1$, $x = 1$, and $x = \sqrt{y}$
The region of integration looks like this:



Since the final integration is toward the *y*-axis between $y = 0$ and $y = 1$, then the other two equations are the right and left boundaries of the region, at , $x = 1$, and $x = \sqrt{y}$ respectively.



In order to rewrite the integral, we need to re-orient the region, so that we are projecting instead toward the *x*-axis at the end of the problem, which gives us limits of $x = 0$ and $x = 1$ for our outermost integral. Now, our innermost integral has limits which form the top and bottom of our region instead. The bottom edge is the *x*-axis itself, so this limit is clearly $y = 0$. The upper limit is the parabolic region which can be written as $x = \sqrt{y}$ or $y = x^2$. Since we need these limits to be in the form "*y* = something" we use the second of the two options.

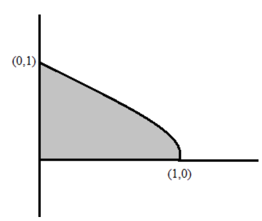
Now our new integral is: $\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \frac{ye^{x^2}}{x^3} dydx$
Since there is only one "y" term in the formula, we focus on that for the first integration:

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \frac{ye^{x^2}}{x^3} dydx = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} y\frac{e^{x^2}}{x^3} dydx = \int_{x=0}^{x=1} \left(\frac{1}{2}y^2 \frac{e^{x^2}}{x^3}\right)\Big|_{y=0}^{y=x^2} dx = \int_{x=0}^{x=1} \left(\frac{1}{2}x^4 \frac{e^{x^2}}{x^3} - 0\right) dx$$

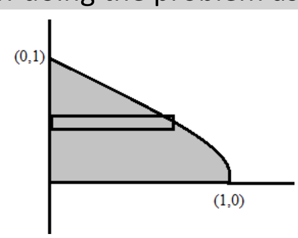
$$\int_{x=0}^{x=1} \left(\frac{1}{2}xe^{x^2}\right) dx = \left(\frac{1}{4}e^{x^2}\right)\Big|_{x=0}^{x=1} = \left(\frac{1}{4}e^1\right) - \left(\frac{1}{4}e^0\right) = \frac{e}{4} - \frac{1}{4} = \frac{e-1}{4}$$

11. Find the mass of a thin metal plate that occupies a region D that is bounded by the parabola $x = 1 - y^2$ and the coordinate axes in the first quadrant if the density of the plate varies according to the density function $\rho(x, y) = y$

The region outlined by the boundaries of
The plate look like this:

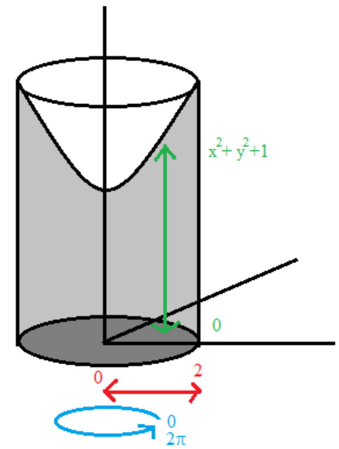


Note that we could set this up either as a double integral that ends along the *x*-axis, or as a double integral that ends along the *y*-axis. If we try the first option, we will have to rewrite that parabola in its alternate form: $x = 1 - y^2 \Longrightarrow y = \sqrt{1 - x}$
This will make our integration slightly messier, so we are probably better off doing the problem as an integration toward the *y*-axis, which will make our region look like this:



Therefore, the equation for the mass will simply be:
$m = \iint [\text{Density}] dxdy = \int_{y=0}^{y=1} \int_{x=0}^{x=1-y^2} ydxdy$
$\int_{y=0}^{y=1} (xy)|_{x=0}^{x=1-y^2} dy = \int_{y=0}^{y=1} y(1 - y^2) - 0dy = \int_{y=0}^{y=1} y - y^3 dy$
$\left(\frac{1}{2}y^2 - \frac{1}{4}y^4\right)\Big|_{y=0}^{y=1} = \left(\frac{1}{2} - \frac{1}{4}\right) - (0 - 0) = \frac{1}{4}$
Thus the total mass of the plate is: $m = \frac{1}{4}$

12. Find the volume that lies inside the cylinder $x^2+y^2=4$ and above the *xy*-plane and beneath the paraboloid $z=x^2+y^2+1$ by using cylindrical coordinates.



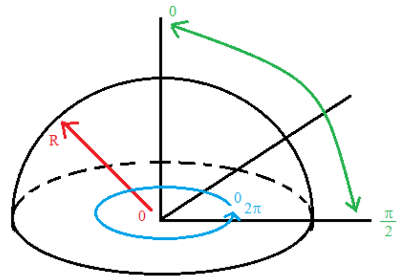
Graphing the region, we can see that it looks like this:
Since the shadow region in the *xy*-plane is a circle,
This gives us very convenient bounds
for our r and θ terms: $0 \le r \le 2$ and $0 \le \theta \le 2\pi$
The *z*-coordinate bounds are the equations that form the upper and lower Edges of this region. The upper surface is the paraboloid, whose equation is $z=x^2+y^2+1$. The lower boundary is the *xy*-plane, whose equation is simply $z=0$.

Thus we can convert the fairly involved rectangular integral: $\int_{x=-2}^{x=2}\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}\int_{z=0}^{z=x^2+y^2+1} 1dzdydx$
Into a much more convenient cylindrical form by using the following transformation equations:
$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2+y^2$
$dxdydz = rdrd\theta dz$

This makes our integral into: $\int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi}\int_{z=0}^{z=r^2+1} 1rdzd\theta dr$
$\int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi} (rz)\big|_{z=0}^{z=1+r^2} d\theta dr = \int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi}(r(1+r^2)-0)d\theta dr = \int_{r=0}^{r=2}\int_{\theta=0}^{\theta=2\pi}(r+r^3)d\theta dr$
$= \int_{r=0}^{r=2}\left((r+r^3)\theta\right)\Big|_{\theta=0}^{\theta=2\pi} dr = \int_{r=0}^{r=2} 2\pi(r+r^3)dr$
$\left(2\pi\left(\frac{1}{2}r^2+\frac{1}{4}r^4\right)\right)\Big|_{r=0}^{r=2} = \left(2\pi(2+4)\right)-(0) = 12\pi$

13. Show that the volume of the upper half of a sphere of radius R is $\frac{2}{3}\pi R^3$ by using spherical coordinates.

Graphing the upper half of a sphere, we can see that
It looks like this:
Thus, we can tell that we have very convenient bounds
For all of our limits: $0 \le \rho \le R$ and $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{2}$



If we had tried to find the volume in rectangular coordinates,
The equations would have been very complicated: $\int_{x=-R}^{x=R} \int_{y=-\sqrt{R^2-x^2}}^{y=\sqrt{R^2-x^2}} \int_{z=-\sqrt{R^2-x^2-y^2}}^{z=\sqrt{R^2-x^2-y^2}} 1dzdydx$
However, to convert to spherical, we simply use the following transformation equations:
$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$
$\rho^2 = x^2 + y^2 + z^2$
$dxdydz = \rho^2 \sin\phi \, d\rho d\theta d\phi$

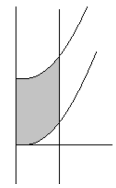
This makes our integral into: $\int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=R} 1\rho^2 \sin\phi \, d\rho d\theta d\phi$
$\int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{3}\rho^3 \sin\phi\right)\Big|_{\rho=0}^{\rho=R} d\theta d\phi = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \left(\frac{1}{3}R^3 \sin\phi\right) d\theta d\phi = \int_{\phi=0}^{\phi=\frac{\pi}{2}} \left(\frac{1}{3}R^3\, \theta \sin\phi\right)\Big|_{\theta=0}^{\theta=2\pi} d\phi$
$\int_{\phi=0}^{\phi=\frac{\pi}{2}} \left(\frac{2\pi}{3}R^3 \sin\phi\right) d\phi = \left(-\frac{2\pi}{3}R^3 \cos\phi\right)\Big|_{\phi=0}^{\phi=\frac{\pi}{2}} = \left(-\frac{2\pi}{3}R^3(0)\right) - \left(-\frac{2\pi}{3}R^3(1)\right) = 0 + \frac{2\pi}{3}R^3 = \frac{2}{3}\pi R^3$

14. Evaluate the following integral: $\int_0^1 \int_{4x^2}^{4x^2+1} 8x\,dy\,dx$

by making the transformations: $s = 2x$ and $t = y - 4x^2$

(Note: You *must* show the appropriate work for a change-of-variable problem. You will not receive any credit if you attempt to leave the integral in its original *xy*-form.)

Now the region of integration looks like this:



Since the integral looks like this: $\int_{x=0}^{x=1} \int_{y=4x^2}^{y=4x^2+1} 8x\,dy\,dx$

It has four boundaries: $x = 1$ [right]

and $x = 0$ [left] and $y = 4x^2$ [bottom] and $y = 4x^2 + 1$ [top]

We are given the equations $s = 2x$ and $t = y - 4x^2$, but we have to convert the problem into *s* and *t*, so we need to solve them the other way: $x = \frac{1}{2}s$

and $t = y - 4x^2 \Rightarrow t = y - s^2 \Rightarrow y = t + s^2$

Now, by using these equations, we can see how the limits transform:

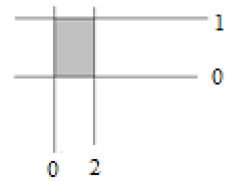
[top] $y = 4x^2 + 1 \Rightarrow t + s^2 = s^2 + 1 \Rightarrow t = +1$

[bottom] $y = 4x^2 \Rightarrow t + s^2 = s^2 \Rightarrow t = 0$

[left] $x = 0 \Rightarrow \frac{1}{2}s = 0 \Rightarrow s = 0$

[right] $x = 1 \Rightarrow \frac{1}{2}s = 1 \Rightarrow s = 2$

This gives us a very simple rectangular region for the new integral:



Now, we need to determine the "extra" factor from the Jacobian:

Since $x = \frac{1}{2}s \Rightarrow \frac{\partial x}{\partial s} = \frac{1}{2}$ and $\frac{\partial x}{\partial t} = 0$

Since $y = t + s^2 \Rightarrow \frac{\partial y}{\partial s} = 2s$ and $\frac{\partial y}{\partial t} = 1$

Then $J = \begin{bmatrix} \frac{1}{2} & 0 \\ 2s & 1 \end{bmatrix}$ so $\det(J) = \left(\frac{1}{2}\right)(1) - (0)(2s) = \frac{1}{2}$

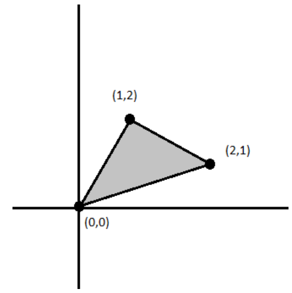
Since $dxdy = |\det(J)|dsdt = \left|\frac{1}{2}\right| dsdt = \frac{1}{2}dsdt$

Finally, we have that $\int_{x=0}^{x=1} \int_{y=4x^2}^{y=4x^2+1} 8x\,dy\,dx = \int_{t=0}^{t=1} \int_{s=0}^{s=2} 8\left(\frac{1}{2}s\right) \cdot \frac{1}{2}dsdt$

$\int_{t=0}^{t=1} \int_{s=0}^{s=2} 2s\,ds\,dt = \int_{t=0}^{t=1} (s^2)|_{s=0}^{s=2} dt = \int_{t=0}^{t=1} 4 - 0dt = \int_{t=0}^{t=1} 4dt$

$(4t)|_{t=0}^{t=1} = 4 - 0 = 4$

15. Evaluate the following integral: $\iint_R (x-2y)dA$ over the triangular region R that has vertices at the points $(0,0)$ and $(1,2)$ and $(2,1)$ by making the transformations: $x = 2u + v$ and $y = u + 2v$
(Note: You *must* show the appropriate work for a change-of-variable problem. You will not receive any credit if you attempt to leave the integral in its original *xy*-form.)



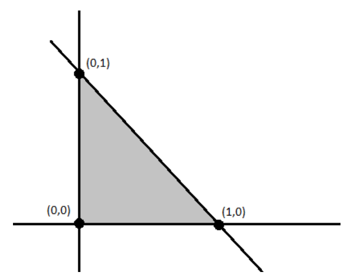
Now the region of integration looks like this:
It has three boundaries: $y = 2x$ and $y = \frac{1}{2}x$ and $y = -x + 3$
This is inconvenient to write in xy-form because no matter which axis we Choose for our final integration, the region would have to be subdivided into Two pieces. We are already given the transformation equations $x = 2u + v$ and $y = u + 2v$, so we convert these edges one at a time:

Edge 1: $y = 2x \Longrightarrow [u + 2v] = 2[2u + v] \Longrightarrow u + 2v = 4u + 2v \Longrightarrow 2v - 2v = 4u - u$
$\Longrightarrow 3u = 0 \Longrightarrow u = 0$

Edge 2: $y = \frac{1}{2}x \Longrightarrow [u + 2v] = \frac{1}{2}[2u + v] \Longrightarrow u + 2v = u + \frac{1}{2}v \Longrightarrow 2v - \frac{1}{2}v = u - u$
$\Longrightarrow \frac{3}{2}v = 0 \Longrightarrow v = 0$

Edge 3: $y = -x + 3 \Longrightarrow [u + 2v] = -[2u + v] + 3 \Longrightarrow u + 2v = -2u - v + 3$
$\Longrightarrow 2v + v = -2u - u + 3 \Longrightarrow 3v = -3u + 3 \Longrightarrow v = -u + 1$

This gives us a convenient region in our new uv-system:



Now, we need to determine the "extra" factor from the Jacobian:
Since $x = 2u + v \Longrightarrow \frac{\partial x}{\partial u} = 2$ and $\frac{\partial x}{\partial v} = 1$
Since $y = u + 2v \Longrightarrow \frac{\partial y}{\partial u} = 1$ and $\frac{\partial y}{\partial v} = 2$
Then $J = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ so $\det(J) = (2)(2) - (1)(1) = 4 - 1 = 3$
Since $dxdy = |\det(J)|dudv = |3|dudv = 3dudv$
Additionally, we need to rewrite the function we are integrating, so
$x - 2y \Longrightarrow [2u + v] - 2[u + 2v] = 2u + v - 2u - 4v \Longrightarrow -3v$
Finally, we have that $\iint_R (x - 2y)dA = \int_{u=0}^{u=1}\int_{v=0}^{v=1-u}(-3v)\cdot 3dvdu$

$$\int_{u=0}^{u=1}\int_{v=0}^{v=1-u} -9vdvdu = \int_{u=0}^{u=1} -\frac{9}{2}(v^2)\Big|_{v=0}^{v=1-u} du = \int_{u=0}^{u=1} -\frac{9}{2}(1-u)^2 + 0du$$
$$= \int_{u=0}^{u=1} -\frac{9}{2}(1-u)^2 du$$
$$= \frac{3}{2}(1-u)^3\Big|_{u=0}^{u=1} = \frac{3}{2}(1-1)^3 - \frac{3}{2}(1-0)^3 = \frac{3}{2}(0)^3 - \frac{3}{2}(1)^3 = 0 - \frac{3}{2} = -\frac{3}{2}$$

16. Evaluate the following line integrals by parameterizing the curves:

$\int_C 8x\,ds$ where C is the arc of the parabola $y = x^2$ from $(0,0)$ to $(2,4)$

Since $y = x^2$, then we can parameterize the curve by letting $x = t$ and $y = t^2$ for $0 \le t \le 2$
Since $x = t \Longrightarrow dx = 1dt$ and $y = t^2 \Rightarrow dy = 2tdt$,
then $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{(1dt)^2 + (2tdt)^2} = \sqrt{1 + 4t^2}dt$
$\int_{t=0}^{t=2} 8xds = \int_{t=0}^{t=2} 8t\sqrt{1 + 4t^2}dt = \int_{t=0}^{t=2} 8t(1 + 4t^2)^{\frac{1}{2}}dt = \left(\frac{2}{3}(1 + 4t^2)^{\frac{3}{2}}\right)\Big|_{t=0}^{t=2}$
$= \frac{2}{3}\left((17)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right) = \frac{2(17\sqrt{17}-1)}{3}$

$\int_C y^3dx + x^2dy$ where C is the arc of the parabola $x = 1 - y^2$ from $(0,-1)$ to $(0,1)$

Since $x = 1 - y^2$, then we can parameterize the curve by letting $y = t$ and $x = 1 - t^2$ for $-1 \le t \le 1$
Since $x = 1 - t^2 \Longrightarrow dx = -2tdt$ and $y = t \Rightarrow dy = 1dt$,
$\int_{t=-1}^{t=1} y^3dx + x^2dy = \int_{t=-1}^{t=1}(t)^3(-2tdt) + (1 - t^2)^2(1dt) = \int_{t=-1}^{t=1}(-2t^4 + (1 - t^2)^2)dt =$
$\int_{t=-1}^{t=1}(-2t^4 + 1 - 2t^2 + t^4)dt = \int_{t=-1}^{t=1}(1 - 2t^2 - t^4)dt = \left(t - \frac{2}{3}t^3 - \frac{1}{5}t^5\right)\Big|_{t=-1}^{t=1}$
$= \left(1 - \frac{2}{3} - \frac{1}{5}\right) - \left(-1 + \frac{2}{3} + \frac{1}{5}\right) = 1 - \frac{2}{3} - \frac{1}{5} + 1 - \frac{2}{3} - \frac{1}{5} = 2 - \frac{4}{3} - \frac{2}{5} = \frac{30}{15} - \frac{20}{15} - \frac{6}{15} = \frac{4}{15}$

$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}} = e^z\mathbf{i} + xz\mathbf{j} + (x + y)\mathbf{k}$ and C is given by $\vec{\mathbf{r}}(t) = t^2\mathbf{i} + t^3\mathbf{j} - t\mathbf{k}$ for $0 \le t \le 1$

Now $\vec{\mathbf{r}} = \langle x, y, z\rangle$ always. Since we are told that $\vec{\mathbf{r}}(t) = t^2\mathbf{i} + t^3\mathbf{j} - t\mathbf{k} = \langle t^2, t^3, -t\rangle$, then we know that we can parameterize the curve by letting $x = t^2$ and $y = t^3$ and $z = -t$ for $0 \le t \le 1$
This means that $\vec{\mathbf{F}}(t) = e^z\mathbf{i} + xz\mathbf{j} + (x + y)\mathbf{k} = \langle e^z, xz, x + y\rangle = \langle e^{-t}, -t^3, t^2 + t^3\rangle$
Furthermore, since $x = t^2 \Longrightarrow dx = 2tdt$ and $y = t^3 \Rightarrow dy = 3t^2dt$ and $dz = -t \Rightarrow dz = -1dt$
then $d\vec{\mathbf{r}} = \langle dx, dy, dz\rangle = \langle 2t, 3t^2, -1\rangle dt$
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{t=0}^{t=1}\langle e^{-t}, -t^3, t^2 + t^3\rangle \cdot \langle 2t, 3t^2, -1\rangle dt = \int_{t=0}^{t=1}(2te^{-t} - 3t^5 - t^2 - t^3)dt$
$= \left(-2te^{-t} - 2e^{-t} - \frac{1}{2}t^6 - \frac{1}{3}t^3 - \frac{1}{4}t^4\right)\Big|_{t=0}^{t=1} = \left(-2e^{-1} - 2e^{-1} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right) - (0 - 2 - 0 - 0 - 0)$
$= -\frac{4}{e} - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + 2 = -\frac{48}{12e} - \frac{6e}{12e} - \frac{4e}{12e} - \frac{3e}{12e} + \frac{24e}{12e} = \frac{11e-48}{12e}$ or $\frac{11}{12} - \frac{4}{e}$

17. Use the Fundamental Theorem of Line Integrals to evaluate the following integral:

$\int_C (2yz + 2x + e^y)dx + (2xz + xe^y + e^z)dy + (2xy + ye^z + \pi\cos(\pi z))dz$

where C is the line segment parameterized by the function $\vec{\mathbf{r}}(t) = 4t\mathbf{i} + (2-2t)\mathbf{j} + 3t\mathbf{k}$ for $0 \le t \le 1$.

Since the Fundamental Theorem of Line Integrals applies to vector fields that are conservative, we should be able to find a scalar function whose gradient is the vector field we are trying to integrate. Since $\int_C Pdx + Qdy + Rdz = \int_C \langle P, Q, R\rangle \cdot \langle dx, dy, dz\rangle = \int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$, then our vector function is:

$\vec{\mathbf{F}} = \langle 2yz + 2x + e^y, 2xz + xe^y + e^z, 2xy + ye^z + \pi\cos(\pi z)\rangle$

Let us see if we can construct a scalar function that will produce this as a gradient.

If we had such a function f, its gradient would be: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\rangle$. So let us try to antidifferentiate the component functions appropriately and see what we come up with:

$\int 2yz + 2x + e^y dx = 2xyz + x^2 + xe^y$

$\int 2xz + xe^y + e^z dy = 2xyz + xe^y + ye^z$

$\int 2xy + ye^z + \pi\cos(\pi z)\, dz = 2xyz + ye^z + \sin(\pi z)$

Now our function is composed of several parts. There can be terms which involve all three variables, some that involve only two, others that have only one, and so on. In the end, it should look something like this: $f = A(x,y,z) + B(x,y) + C(y,z) + D(x,z) + E(x) + F(y) + G(z)$

Comparing this to our antiderivatives, we get: $f = 2xyz + xe^y + ye^z + 0 + x^2 + 0 + \sin(\pi z)$

And so we have shown that $\vec{\mathbf{F}}$ is conservative and that it is the gradient of the potential function $f = 2xyz + xe^y + ye^z + x^2 + \sin(\pi z)$

The Fundamental Theorem of Line Integrals tells us that to integrate a conservative function, we simply need to evaluate the potential function at the endpoints of the path. The actual path itself does not matter, only the starting and ending positions do. Thus, we look at the parameterized curve C and determine where the endpoints of the path are:

Since $\vec{\mathbf{r}}(t) = 4t\mathbf{i} + (2-2t)\mathbf{j} + 3t\mathbf{k}$ and $0 \le t \le 1$,

then at $t = 0$ we get: $\vec{\mathbf{r}}(0) = 0\mathbf{i} + (2-0)\mathbf{j} + 0\mathbf{k} = \langle 0,2,0\rangle$

and at $t = 1$ we get: $\vec{\mathbf{r}}(1) = 4\mathbf{i} + (2-2)\mathbf{j} + 3\mathbf{k} = \langle 4,0,3\rangle$

We evaluate the potential function at these endpoints, which gives us:

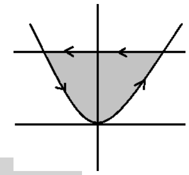
$f(4,0,3) = 2(4)(0)(3) + 4e^0 + 0e^3 + 4^2 + \sin(3\pi) = 0 + 4 + 0 + 16 + 0 = 20$

$f(0,2,0) = 2(0)(2)(0) + 0e^2 + 2e^0 + 0^2 + \sin(0) = 0 + 0 + 2 + 0 + 0 = 2$

Thus we know that: $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{t=a}^{t=b} \nabla f \cdot d\vec{\mathbf{r}} = f|_{t=a}^{t=b} = f(b) - f(a) = 20 - 2 = 18$

18. Use Green's Theorem to evaluate the integral $\int_C 2x^2y^2dx - x^3ydy$ where C is the arc of the parabola $y = x^2$ from $(-1,1)$ to $(1,1)$ and then a line connecting $(1,1)$ to $(-1,1)$.

Now the region looks like this:



It is obviously a loop, so we apply Green's Theorem and rewrite the integral as:

$\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{x=-1}^{x=1}\int_{y=x^2}^{y=1}(-3x^2y - 4x^2y)dydx$

$\int_{x=-1}^{x=1}\int_{y=x^2}^{y=1} -7x^2ydydx = \int_{x=-1}^{x=1}\left(-\frac{7}{2}x^2y^2\right)\Big|_{y=x^2}^{y=1} dx = \int_{x=-1}^{x=1} -\frac{7}{2}x^2(1-x^4)dx$

$\int_{x=-1}^{x=1}\left(-\frac{7}{2}x^2 + \frac{7}{2}x^6\right)dx = \left(-\frac{7}{6}x^3 + \frac{1}{2}x^7\right)\Big|_{x=-1}^{x=1} = \left(-\frac{7}{6} + \frac{1}{2}\right) - \left(\frac{7}{6} - \frac{1}{2}\right) = -\frac{7}{6} + \frac{1}{2} - \frac{7}{6} + \frac{1}{2} = -\frac{14}{6} + \frac{2}{2}$

$-\frac{7}{3} + 1 = -\frac{7}{3} + \frac{3}{3} = -\frac{4}{3}$

19. Use Stokes' Theorem to evaluate the integral $\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ where $\vec{\mathbf{F}} = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$ and C is the triangle with vertices at $(1,0,0)$ and $(0,1,0)$ and $(0,0,1)$ with counterclockwise orientation when viewed from above.

The instructions tell us to use the Stokes' Theorem. So we need to rewrite the problem:
$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S \mathrm{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}}$, which we can do since C is a closed loop.
(This also prevents us from having to deal with a line integral that is made of three segments, which we would have to split into three separate problems and parameterize separately, which would be tedious. Stokes' Theorem lets us write everything as a single double integral, which would be preferable.)
We begin by calculating $\mathrm{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \vec{\mathbf{F}}$

$$\mathrm{curl}\,(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \times \langle xy, yz, xz \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix}$$

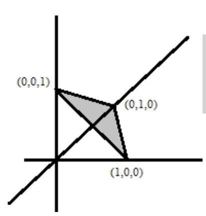
$= (0 - y)\mathbf{i} - (z - 0)\mathbf{j} + (0 - x)\mathbf{k} = -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}$
Thus $\mathrm{curl}\,(\vec{\mathbf{F}}) = \langle -y, -z, -x \rangle$

Now we need to parameterize the surface.
The surface is a part of the plane $x + y + z = 1$, which can be rewritten as $z = 1 - x - y$
Thus, we can write points on the surface as:
$\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle x, y, 1 - x - y \rangle$
Thus we see that we can parameterize our formula purely in terms of the variables *x* and *y*, eliminating *z*.
This allows us to rewrite our function as: $\mathrm{curl}\,(\vec{\mathbf{F}}) = \langle -y, -z, -x \rangle = \langle -y, -1 + x + y, -x \rangle$
We also need to make $d\vec{\mathbf{S}} = \vec{\mathbf{r}}_u \times \vec{\mathbf{r}}_v = \vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y$
Now since $\vec{\mathbf{r}} = \langle x, y, z \rangle = \langle x, y, 1 - x - y \rangle$ then we know that $\vec{\mathbf{r}}_x = \langle 1,0,-1 \rangle$ and $\vec{\mathbf{r}}_y = \langle 0,1,-1 \rangle$

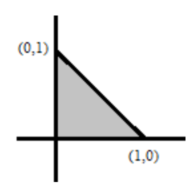
$$\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (0+1)\mathbf{i} - (-1-0)\mathbf{j} + (1-0)\mathbf{k}$$

$\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y = \langle 1,1,1 \rangle$
Observe that this vector points *upward* (as we would expect from the counterclockwise orientation) and we are ready to move on.
Thus $\mathrm{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \mathrm{curl}(\vec{\mathbf{F}}) \cdot (\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y)$
$= \langle -y, -1 + x + y, -x \rangle \cdot \langle 1,1,1 \rangle$
$= -y - 1 + x + y - x$
$= -1$
And all we need to do now is find our limits to be able to finish the problem.

Now our surface looks like this:



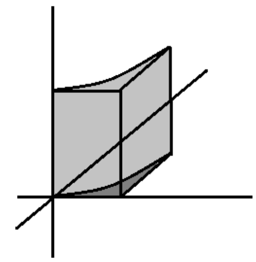
Which lives above a region in *xy*-plane that looks like this:



This means that our integral is really: $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}} \cdot (\vec{\mathbf{r}}_x \times \vec{\mathbf{r}}_y)\, dydx$
We can orient this integration so that it ends along the *x*-axis with limits of $x = 0$ and $x = 1$, which means that our *y*-limits will be the upper boundary which is the line $y = 1 - x$ and the lower boundary is $y = 0$
So: $\iint_S \mathrm{curl}(\vec{\mathbf{F}}) \cdot d\vec{\mathbf{S}} = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} -1 dydx$
$\int_{x=0}^{x=1} (-y)\big|_{y=0}^{y=1-x} dx = \int_{x=0}^{x=1} (x - 1) - (0) dx = \int_{x=0}^{x=1} x - 1 dx$
$= \left(\frac{1}{2}x^2 - x\right)\Big|_{x=0}^{x=1} = \left(\frac{1}{2} - 1\right) - (0 - 0) = -\frac{1}{2}$

20. Use the Divergence Theorem to evaluate the surface integral $\iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} dS$ (which can also be written as $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}}$) where $\vec{\mathbf{F}} = (5x + 2xy)\,\mathbf{i} + (4xz - y^2)\,\mathbf{j} + (e^x - 3z)\,\mathbf{k}$ and S is the surface bounded by the parabolic cylinder $y = x^2$ and the planes $y = 0$ and $x = 1$ and $z = 3$ and and $z = 0$ with outward orientation

The instructions tell us to use the Divergence Theorem. So we need to rewrite the problem: $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_V \operatorname{div}(\vec{\mathbf{F}})\, dV$, which we can do since S is a closed surface which looks like this:



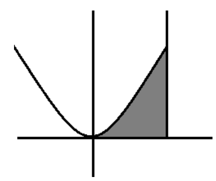
(This also prevents us from having to deal with a surface integral that is made of multiple edges. This particular one has five different edge surfaces, which we would have to split into separate problems, parameterizing each, which would take a very long time.
The Divergence Theorem lets us write everything as a single triple integral, which is a tremendous improvement.)

We begin by calculating $\operatorname{div}(\vec{\mathbf{F}}) = \nabla \cdot \vec{\mathbf{F}} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \vec{\mathbf{F}}$

$\operatorname{div}(\vec{\mathbf{F}}) = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle 5x + 2xy, 4xz - y^2, e^x - 3z \rangle = 5 + 2y - 2y - 3 = 2$

This means that our integral is now: $\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iiint_V 2\, dxdydz$

Finding the limits of integration in *xyz*-form is not particularly difficult, since the region projects down into a "shadow" region beneath it in the *xy*-plane that looks like this:



Thus our integral becomes: $\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=3} 2dzdydx$

$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} \int_{z=0}^{z=3} 2dzdydx$
$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} (2z)|_{z=0}^{z=3} dydx$
$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} (6) - (0) dydx$
$\int_{x=0}^{x=1} \int_{y=0}^{y=x^2} 6dydx$
$\int_{x=0}^{x=1} (6y)|_{y=0}^{y=x^2}\, dx$
$\int_{x=0}^{x=1} (6x^2) - (0) dx$
$\int_{x=0}^{x=1} 6x^2 dx$
$= (2x^3)|_{x=0}^{x=1} = 2 - 0 = 2$