# The Three-Dimensional Spherical Quantum Harmonic Oscillator: A Theoretical Exploration


Aurora M. Bowman

Faculty Advisor: Dr. Souvik Das, Department of Aerospace, Physics, & Space Science, Florida Institute of Technology


## ABSTRACT


- The three-dimensional quantum harmonic oscillator is analyzed in spherical coordinates.
- The Schrödinger equation is solved using separation of variables.
- The system exhibits a quantized energy spectrum determined by quantum numbers.
- Results describe the structure of radial and angular wavefunctions.


## INTRODUCTION & BACKGROUND

- The quantum harmonic oscillator is a fundamental model in quantum mechanics.
- The time-independent 3-D Schrödinger equation in Cartesian coordinates is given by

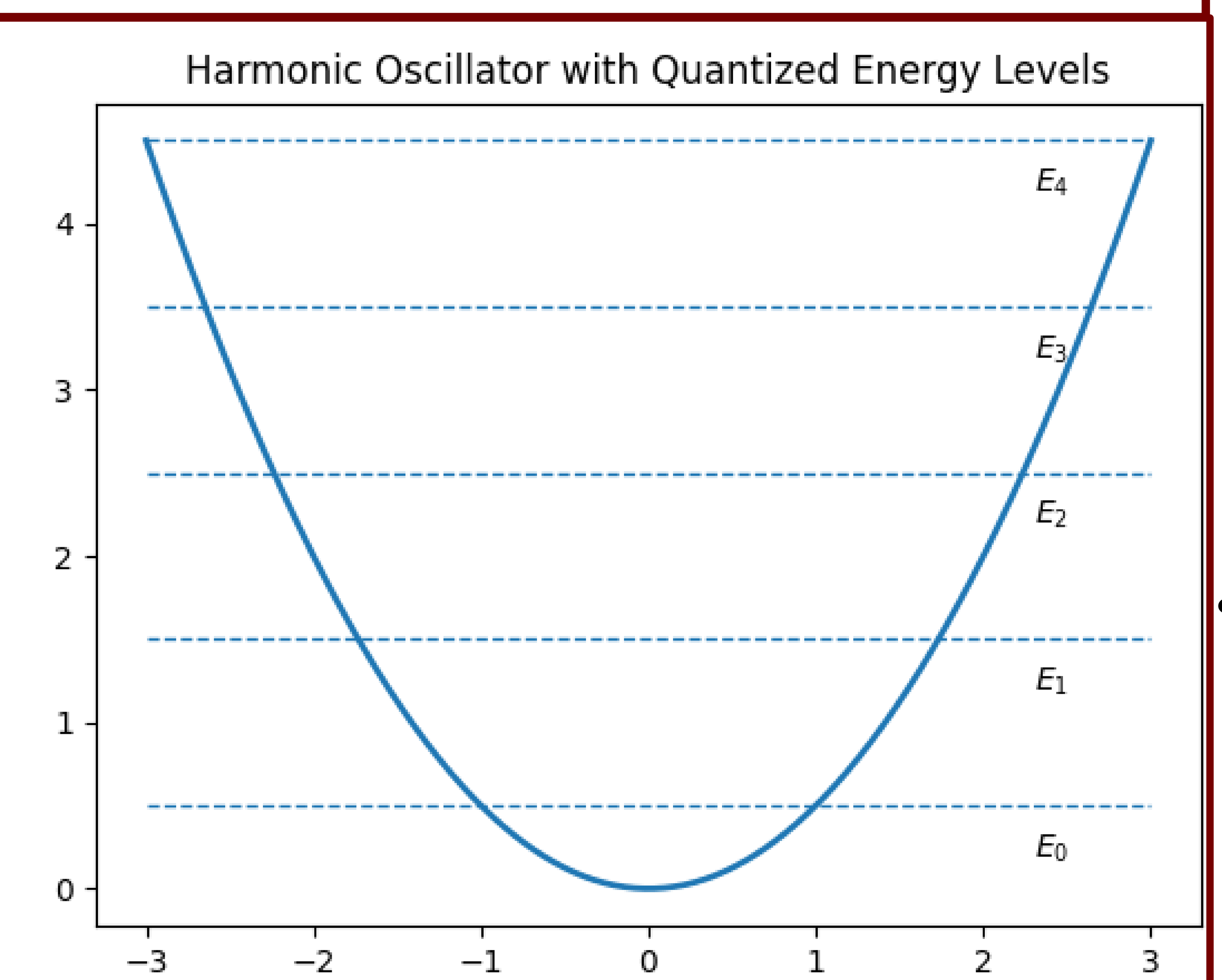
$$\frac{-\hbar^2}{2m}\nabla^2\Psi + V(x,y,z)\Psi = E\Psi.$$

- The potential is given by

$$V(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2)$$

where $\omega$ is angular frequency and $m$ is the particle's mass.

- In 3-D spherical symmetry, spherical coordinates are most natural.
- Solving the Schrödinger equation leads to discrete energy levels.




## REFERENCES


Brandt et al., *Special Functions of Mathematical Physics*, Springer (2003).

Yan & Huang, *Visualization and Analysis of 3D Linear Harmonic Oscillator*, AJMS (2023).


## METHODOLOGY

- Begin with the time-independent Schrödinger equation.
- Expressing the Laplacian $\nabla^2\Psi$ in spherical coordinates yields:

$$\frac{-\hbar^2}{2m}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Psi}{\partial\theta}\right)+\frac{1}{r^2 sin^2(\theta)}\frac{\partial^2\Psi}{\partial\phi^2}\right)+V(r)\Psi = E\Psi$$

with potential $V(r)=\frac{1}{2}m\omega^2r^2$.

- Apply separation of variables to decouple radial and angular differential equations and solve each part individually.

## RESULTS

- Separation of variables yields a solution of the form

$$\boldsymbol{\Psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)}$$

with angular and radial solutions

$$\boldsymbol{\Phi(\phi) = \frac{1}{\sqrt{2\pi}}e^{im\phi}}$$

$$\boldsymbol{\Theta(\theta) \propto P_\ell^{|m|}(cos\ \theta)}$$

$$\boldsymbol{R(r) = N_k r^\ell e^{\frac{-r^2}{2a^2}} L_k^{\ell+\frac{1}{2}}\left(\frac{r^2}{a^2}\right)}$$

where the normalization factor $N_k = \sqrt{\frac{2k!}{a^3\Gamma(k+\ell+\frac{3}{2})}}$

and associated Laguerre polynomials are defined as

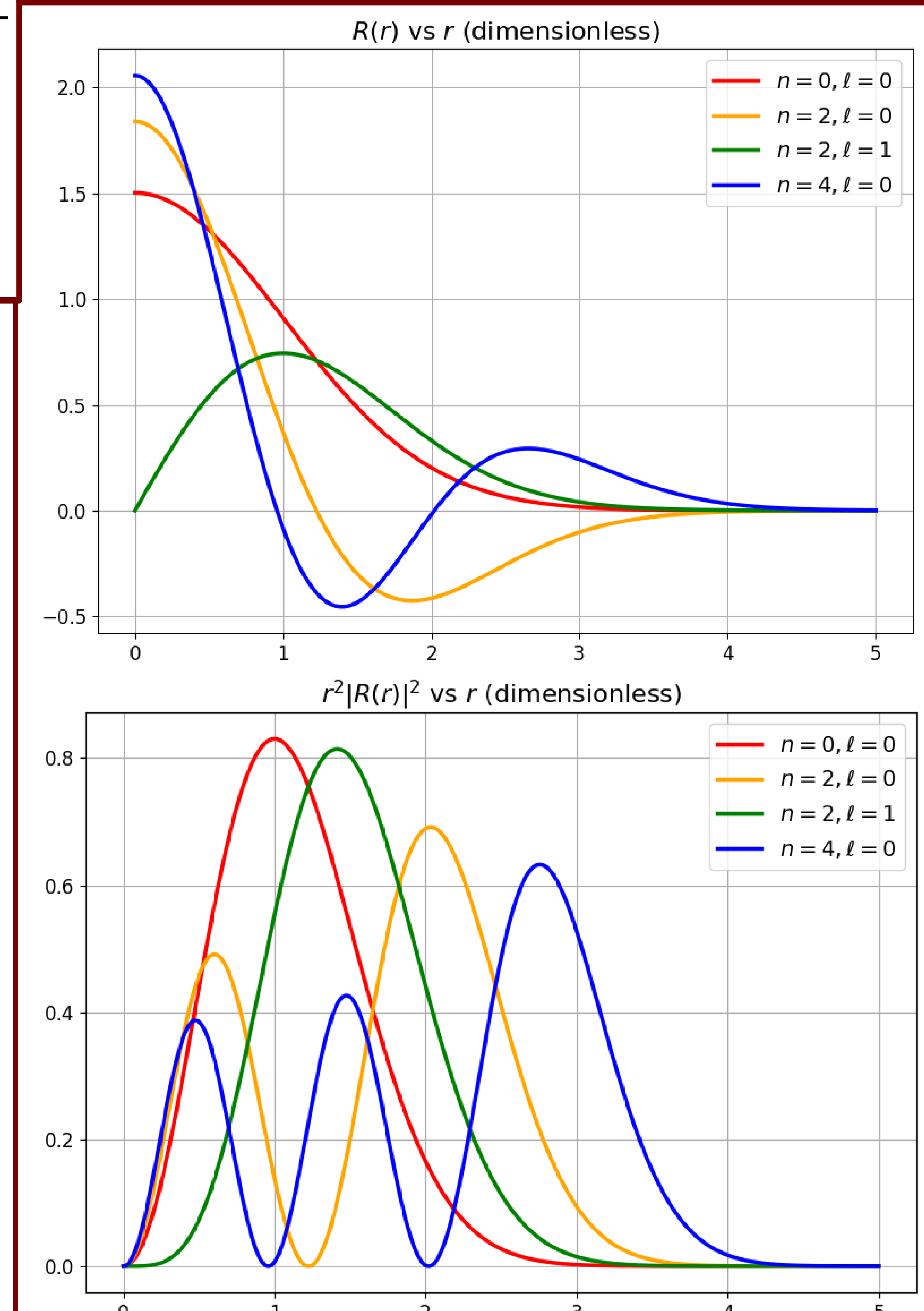
| $k$ | $L_k(x)$ with $x=\frac{r^2}{a^2}$ |
|---|---|
| 0 | $-1$ |
| 1 | $-x+1$ |
| 2 | $\frac{1}{2}(x^2-4x+2)$ |

- Quantum numbers arise from boundary conditions and define the energy, angular momentum, and orientation of each state:

| | |
|---|---|
| Magnetic ($m$) | $m\in\mathbb{Z},\quad \|m\|\le\ell$ |
| Angular momentum ($\ell$) | $\ell = n, n-2, n-4, \ldots \ge 0$ |
| Radial ($k$) | $k=\frac{n-\ell}{2}$ |
| Total ($n$) | $n=2k+\ell,\quad n=0,1,2\ldots$ |

- Energy quantization arises from requiring normalizable solutions to the radial equation:

$$\boldsymbol{E_n = \hbar\omega(n+\tfrac{3}{2})}.$$




Plots of radial wavefunctions (top) and their probability densities (bottom) created via Python.

## CONCLUSIONS

- $\Theta(\theta)\Phi(\phi)$ determines the wavefunction's shape & orientation in space.
- $R(r)$ determines how the wavefunction varies with distance from the origin.
- Quantum numbers define energy, angular momentum, & structure.
- Higher energy states have more nodes and larger spatial extent.
- Radial probability shifts outward with increasing energy.